

# HOMOTOPY THEORY OF COMPLETE DIFFERENTIAL GRADED LIE ALGEBRAS

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**ABSTRACT.** In a previous work, we associated a complete differential graded Lie algebra model to any finite simplicial complex in a functorial way. We introduce here a closed model category structure on the category of complete differential graded Lie algebras. By determining an explicit cylinder object in this category, we are able to show that two homotopic maps between finite simplicial complexes produce homotopic models. This cylinder provides also an inductive process for the construction of models of Euclidean simplices and, therefore, of any finite simplicial complex. A particular model of Riemannian surfaces is also presented.

In this paper we deal with complete differential graded Lie algebras (cDGL in short) defined over  $\mathbb{Q}$ , and in particular with cDGL that are free as complete Lie algebra (free cDGL in short), i.e., cDGL's of the form  $(\widehat{\mathbb{L}}(V), d)$ , where

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}^{>n}(V).$$

The first known example of this type is the Lawrence-Sullivan interval (a model for the interval) which has the following shape,

$$\mathcal{L}_{[0,1]} = (\widehat{\mathbb{L}}(a, b, x), \partial).$$

Here,  $a$  and  $b$  are Maurer-Cartan elements (cf. Definition 2.1),  $x$  is an element of degree 0 and

$$\partial x = \mathrm{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \mathrm{ad}_x^n(b - a) = \mathrm{ad}_x b + \frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x} - 1}(b - a), \quad (1)$$

where  $B_n$  is the  $n^{\mathrm{th}}$  Bernoulli number.

In [1], we have extended this construction to any simplex  $\Delta^n$ . More generally, to each finite simplicial complex  $X$ , we have associated a free cDGL,  $\mathcal{L}_X$ , in a natural way. Each vertex,  $v_a$ , of  $X$  corresponds to a Maurer-Cartan element,  $a$ , in  $\mathcal{L}_X$ . Moreover,  $\mathcal{L}_X$  equipped with a perturbed differential  $d_a = d + \mathrm{ad}_a$  is a rational model of  $X$ . In particular, if  $X$  is simply connected then  $(\mathcal{L}_X, d_a)$  is quasi-isomorphic to the Quillen rational model of  $X$ . In any cDGL, we interpret a Maurer-Cartan element as a vertex and a cDGL map  $f: \mathcal{L}_{[0,1]} \rightarrow (L, d)$  as a path from  $f(a)$  and to  $f(b)$ .

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In this work, we define a notion of homotopy in the category **cDGL** of complete differential graded Lie algebras,  $(L, d)$  with  $H_*(L, d) = H_{\geq 0}(L, d)$ , for which the following holds.

**Theorem A.** *If  $f, g: X \rightarrow Y$  are two homotopic maps of simplicial complexes, then their models,  $\mathcal{L}_f, \mathcal{L}_g: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ , are homotopic in **cDGL**.*

The key point for this result is a construction of a cylinder made by one of the authors in [13]. If  $(L, d)$  is a free cDGL, its cylinder is a cDGL,  $\text{Cyl}(L, d)$ , equipped with two injections,  $\iota_0, \iota_1: L \rightarrow \text{Cyl}(L)$ , and a projection,  $p: \text{Cyl}(L) \rightarrow L$ . We first look at its behaviour in a simple case.

**Theorem B.** *Let  $(L, d) = (\widehat{\mathbb{L}}(a), d)$  be a free cDGL on a Maurer-Cartan element  $a$ . Then  $\text{Cyl}(L, d)$  is isomorphic to the Lawrence-Sullivan interval  $\mathcal{L}_{[0,1]}$ .*

Following this approach, two Maurer-Cartan elements  $u$  and  $v$  in a cDGL,  $(L, d)$ , are called left equivalent if there is a path in  $L$  starting at  $u$  and ending at  $v$ . They are right equivalent if there is a Maurer-Cartan element  $x \in (L, d) \widehat{\otimes} \wedge(t, dt)$  with  $\varepsilon_0(x) = u$  and  $\varepsilon_1(x) = v$ .

**Theorem C** ([7, Theorem 5.5] and [3, Proposition 3.1]). *Two Maurer-Cartan elements are left equivalent if, and only if, they are right equivalent.*

Next, we introduce a closed model structure on **cDGL**, compatible with all of the above. Taking in account Lemma 1.4 and Proposition 4.5 concerning path liftings, surjective maps are good candidates for fibrations. We have also to define weak-equivalences. The quasi-isomorphisms, used in all known model structures in graded abelian categories (see [5] for instance), do not fit, due to the following result.

**Theorem D.** *Let  $X$  be a connected finite simplicial complex, then  $H(\mathcal{L}_X) = 0$ .*

Recall from [1] that the topological meaning of the algebraic invariants of  $\mathcal{L}_X$  are obtained with the perturbed differentials,  $d_a$ . For instance, the group  $H_0(\mathcal{L}_X, d_a)$  is the Malcev completion of the fundamental group of the component of the vertex in  $X$  corresponding to  $a$ , see [1, Theorem 9.1]. Therefore, we consider these perturbed differentials in the definition of weak-equivalences and prove the following result.

**Theorem E.** *Define a map,  $f: (L', d) \rightarrow (L, d)$  in **cDGL** to be*

- *a weak-equivalence if it is a bijection between the classes of Maurer-Cartan elements and a quasi-isomorphism,  $(L, d_a) \rightarrow (L', d_{f(a)})$ , for any perturbed differential  $d_a = d + \text{ad}_a$  where  $a$  browses the set of Maurer-Cartan elements,*
- *a fibration if it is a surjection,*
- *a cofibration if it has the LLP with respect to all trivial fibrations.*

*Then **cDGL** is a closed model category.*

The cylinder introduced above is the right one for this structure of closed model category.

**Theorem F.** *Let  $(L, d)$  in **cDGL** with  $L \cong \widehat{\mathbb{L}}(V)$ . The cylinder construction  $\text{Cyl}(L, d)$  is a cylinder object in the closed model category **cDGL**.*

The cylinder construction gives also a new way to construct the cDGL  $\mathcal{L}_{\Delta^n}$  and, by consequence, the cDGL  $\mathcal{L}_X$ , for any finite simplicial complex,  $X$ . For that, we define the cone as a quotient of the cylinder construction, see Definition 6.9.

**Theorem G.** *The model of  $\Delta^n$  is isomorphic to the cone on the model of  $\Delta^{n-1}$ .*

Our program is carried out in the next sections, whose headings are self-explanatory.

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For the reader interested in one specific result, we mention the correspondence between the theorems stated in the introduction and results along the text:

Theorem A=Theorem 7.1, Theorem B=Theorem 1.2, Theorem C=Theorem 2.3, Theorem D=Theorem 4.1, Theorem E=Theorem 5.1, Theorem G=Theorem 6.10.

### 1. THE LAWRENCE-SULLIVAN INTERVAL

In the Lawrence-Sullivan interval  $\mathcal{L}_{[0,1]} = (\widehat{\mathbb{L}}(a, b, x), \partial)$  the differential  $\partial$  can be written as

$$\begin{aligned} \partial a &= -\frac{1}{2}[a, a] \quad \text{and} \quad \partial b = -\frac{1}{2}[b, b], \\ \partial x &= \text{ad}_x b + \frac{\text{ad}_x}{e^{\text{ad}_x} - 1}(b - a) = \text{ad}_x a + \frac{\text{ad}_{-x}}{e^{-\text{ad}_x} - 1}(b - a), \end{aligned} \quad (2)$$

where 1 denotes the identity map.

In this text we will use the properties of the Lawrence-Sullivan interval, in particular the following proposition, where we denote  $x * y$  the Baker-Campbell-Hausdorff product of two elements  $x$  and  $y$  of degree 0.

**Proposition 1.1.** [6, Theorem 2]. *Let  $(\widehat{\mathbb{L}}(a, b, x), \partial)$ ,  $(\widehat{\mathbb{L}}(a_0, a_1, x_1), \partial)$  and  $(\widehat{\mathbb{L}}(a_1, a_2, x_2), \partial)$  be three Lawrence-Sullivan intervals. The morphism,*

$$\varphi: (\widehat{\mathbb{L}}(a, b, x), \partial) \rightarrow (\widehat{\mathbb{L}}(a_0, a_1, a_2, x_1, x_2), \partial),$$

*defined by  $\varphi(a) = a_0$ ,  $\varphi(b) = a_2$  and  $\varphi(x) = x_1 * x_2$  is a cDGL morphism.*

Now we form the cDGL  $(\widehat{\mathbb{L}}(a, u, su), d)$  where

$$|u| = |a| = -1, \quad |su| = 0, \quad da = -\frac{1}{2}[a, a], \quad du = 0 \quad \text{and} \quad d(su) = u.$$

**Theorem 1.2.** *The morphism*

$$\psi: (\widehat{\mathbb{L}}(a, b, x), \partial) \rightarrow (\widehat{\mathbb{L}}(a, u, su), d)$$

*defined by*

$$\psi(a) = a, \psi(x) = su, \psi(b) = e^{\text{ad}_{-su}}(a) + \frac{e^{\text{ad}_{-su}} - 1}{\text{ad}_{-su}}(u),$$

*is a cDGL isomorphism.*

*Proof.* We construct a derivation of degree 1 on  $\widehat{\mathbb{L}}(a, u, su)$  by  $i(a) = su, i(u) = i(su) = 0$  and a derivation,  $\theta$ , of degree 0 by

$$\theta = i \circ d + d \circ i.$$

From the definition of  $\theta$ , we get  $\theta(u) = \theta(su) = 0$ , and

$$\theta(a) = (i \circ d + d \circ i)(a) = dsu - \frac{1}{2}i[a, a] = u - \text{ad}_{su}(a) = u + \text{ad}_{-su}(a).$$

We deduce by induction,

$$\theta^n(a) = \text{ad}_{-su}^{n-1}(u) + \text{ad}_{-su}^n(a).$$

Therefore

$$e^\theta(a) = \sum_{i \geq 0} \frac{\text{ad}_{-su}^i(a)}{i!} + u \sum_{i \geq 0} \frac{\text{ad}_{-su}^i(u)}{(i+1)!} = e^{\text{ad}_{-su}}(a) + \frac{e^{\text{ad}_{-su}} - 1}{\text{ad}_{-su}}(u) = \psi(b). \quad (3)$$

Formula (3) can be reversed and gives

$$u = \frac{\text{ad}_{-su}}{e^{\text{ad}_{-su}} - 1} e^\theta(a) - \frac{\text{ad}_{-su} e^{\text{ad}_{-su}}}{e^{\text{ad}_{-su}} - 1}(a). \quad (4)$$

In particular, we obtain the surjectivity of  $\psi$  from

$$u = \psi \left( \frac{\text{ad}_{-x}}{e^{\text{ad}_{-x}} - 1} (b - a) + \text{ad}_x(a) \right) = \psi(\partial x), \quad (5)$$

and  $d\psi(x) = \psi(\partial x)$  □

*Remark 1.3.* (1) The inverse isomorphism,  $\psi^{-1}: (\widehat{\mathbb{L}}(a, u, su), d) \rightarrow (\widehat{\mathbb{L}}(a, b, x), \partial)$ , is given by

$$\psi^{-1}(a) = a, \psi^{-1}(su) = x, \psi^{-1}(u) = \partial x.$$

- (2) The differential  $\partial$  of  $\mathcal{L}_{[0,1]}$  is entirely determined by the differential of  $a, b$  and the linear part of the differential of  $x$ , see [1, Theorem 1.4], [8].
- (3) An isomorphism between the enveloping algebra of the Lawrence-Sullivan model for the interval and the enveloping algebra of the cylinder construction has been described in [3, Theorem 3.3].

A cDGL map,  $f: \mathcal{L}_{[0,1]} \rightarrow (L, d)$ , is interpreted as a path from  $f(a)$  to  $f(b)$ . The next result shows that such path admits a lift through surjective cDGL's maps. Surjective maps are thus candidates for playing the role of fibrations in the category **cDGL**, as this will be made precise in Section 5.

**Lemma 1.4.** (*Lifting Lemma*) Let  $p: (L_1, d) \rightarrow (L_2, d)$  be a surjective cDGL morphism. Suppose that  $c \in L_1$  is a Maurer-Cartan element, and that  $f: \mathcal{L}_{[0,1]} \rightarrow (L_2, d)$  is a cDGL morphism with  $f(a) = p(c)$ . Then there is a cDGL morphism  $h: \mathcal{L}_{[0,1]} \rightarrow (L_1, d)$  with  $h(a) = c$  and  $p \circ h = f$ .

$$\begin{array}{ccc} & & (L_1, d) \\ & \nearrow h & \downarrow p \\ \mathcal{L}_{[0,1]} & \xrightarrow{f} & (L_2, d) \end{array}$$

*Proof.* Since  $p$  is surjective, there is an element  $y \in L_1$  with  $p(y) = f(x)$ . With the notation in Theorem 1.2, we define a map  $\rho: (\widehat{\mathbb{L}}(a, u, su), d) \rightarrow (L_1, d)$  by  $\rho(a) = c$ ,  $\rho(su) = y$  and  $\rho(u) = dy$ . Clearly  $p \circ \rho = f \circ \psi^{-1}$ . We set  $h = \rho \circ \psi: \mathcal{L}_{[0,1]} \rightarrow (L_1, d)$ , and get  $p \circ h = f$ .  $\square$

## 2. MAURER-CARTAN ELEMENTS

**Definition 2.1.** An element  $a$  of degree  $-1$  in a cDGL,  $(L, d)$ , is a *Maurer-Cartan element* (or MC-element in short) if  $da = -\frac{1}{2}[a, a]$ . In this case, the derivation  $d_a := d + \text{ad}_a$  is a differential, called the *differential  $d$  perturbed by  $a$* .

Recall that the completed tensor product of a cDGL,  $(L, d)$ , with a commutative differential algebra,  $(A, d_A)$ , is a cDGL,  $(L, d) \widehat{\otimes} (A, d_A)$ , in which the differential is the classical differential of a tensor product and the bracket mixes the bracket on  $L$  and the algebra product on  $A$ ,

$$[x \otimes a, y \otimes b] = (-1)^{|a||y|} [x, y] \otimes ab.$$

We will use this construction with the free commutative differential algebra,  $\wedge(t, dt)$ ,  $|t| = 0$ ,  $|dt| = -1$ . We denote by  $\varepsilon_0, \varepsilon_1: (L, d) \widehat{\otimes} \wedge(t, dt) \rightarrow (L, d)$  the cDGL maps defined by  $\varepsilon_0(t) = 0$ ,  $\varepsilon_1(t) = 1$ ,  $\varepsilon_0(dt) = \varepsilon_1(dt) = 0$  and the identity map on  $L$ .

**Definition 2.2.** Let  $y, z$  be two Maurer-Cartan elements in a cDGL,  $(L, d)$ .

- (1) The MC-elements  $y$  and  $z$  are *left equivalent*,  $y \sim_L z$ , if there exists a morphism  $f: \mathcal{L}_{[0,1]} \rightarrow (L, d)$  with  $f(a) = y$  and  $f(b) = z$ .
- (2) The MC-elements  $y$  and  $z$  are *right equivalent*,  $y \sim_R z$ , if there exists a morphism  $g: (\mathbb{L}(w), \partial) \rightarrow (L, d) \widehat{\otimes} \wedge(t, dt)$  with  $w$  a Maurer-Cartan element,  $\varepsilon_0 \circ g(w) = y$  and  $\varepsilon_1 \circ g(w) = z$ .

By Proposition 1.1, the existence of a path between two left equivalent Maurer-Cartan elements gives an equivalence relation which is the algebraic version of points being in the same path-connected component. We prove now that both equivalence relations give the same class, a result already present in [7, Theorem 5.5] and [3, Proposition 3.1]. Henceforth the set of equivalence classes of Maurer-Cartan elements in a cDGL,  $(L, d)$ , is denoted  $\widetilde{MC}(L, d)$ .

**Theorem 2.3.** In a cDGL,  $(L, d)$ , two Maurer-Cartan elements are left equivalent if, and only if, they are right equivalent.

*Proof.* Let  $(\mathbb{L}(a), d)$  be a cDGL with  $a$  a MC-element. We extend it to  $(\mathbb{L}(a, u, su), d)$  with  $|u| = -1$ ,  $|su| = 0$ ,  $du = 0$ ,  $d(su) = u$ . A derivation  $\bar{\imath}$ , of degree 1, is defined on  $(\widehat{\mathbb{L}}(a, u, su), d) \widehat{\otimes} \wedge (t, dt)$  by  $\bar{\imath}(u) = \bar{\imath}(su) = 0$  and  $\bar{\imath}(a) = t su$ . The derivation  $\bar{\theta} = d\bar{\imath} + \bar{\imath}d$  satisfies  $\bar{\theta}(u) = \bar{\theta}(su) = 0$ ,

$$\bar{\theta}(a) = tu + dt su + t \operatorname{ad}_{-su} a$$

and for  $n \geq 2$ ,

$$\bar{\theta}^n(a) = t^n \operatorname{ad}_{-su}^{n-1}(u) + t^n \operatorname{ad}_{-su}^n(a).$$

Therefore,

$$e^{\bar{\theta}}(a) = dt su + e^{\operatorname{ad}_{-tsu}}(a) + \frac{e^{\operatorname{ad}_{-tsu}} - 1}{\operatorname{ad}_{-su}}(u). \quad (6)$$

The morphism,

$$\bar{\psi}: (\widehat{\mathbb{L}}(a, b, x), \partial) \widehat{\otimes} \wedge (t, dt) \rightarrow (\widehat{\mathbb{L}}(a, u, su), d) \widehat{\otimes} \wedge (t, dt),$$

defined by  $\bar{\psi}(a) = a$ ,  $\bar{\psi}(b) = e^{\bar{\theta}}(a)$  and  $\bar{\psi}(x) = t su$ , is a cDGL morphism. We consider then the composition,

$$\Phi: (\mathbb{L}(w), \partial) \xrightarrow{j} \mathcal{L}_{[0,1]} \widehat{\otimes} \wedge (t, dt) \xrightarrow{\bar{\psi}} (\widehat{\mathbb{L}}(a, u, su), d) \widehat{\otimes} \wedge (t, dt) \xrightarrow{\psi^{-1} \otimes \operatorname{id}} \mathcal{L}_{[0,1]} \widehat{\otimes} \wedge (t, dt),$$

where  $w$  is a MC-element,  $j(w) = b$  and  $\psi^{-1}$  is described in Remark 1.3. A direct computation shows that

$$\Phi(w) = (dt)x + e^{\operatorname{ad}_{-tx}}(a) + \frac{e^{\operatorname{ad}_{-tx}} - 1}{\operatorname{ad}_{-x}}(\partial x). \quad (7)$$

From (2) and (6), we deduce

$$\varepsilon_0 \circ \Phi(w) = a \quad \text{and} \quad \varepsilon_1 \circ \Phi(w) = b.$$

(1) Suppose that  $y$  and  $z$  are left equivalent MC-elements in the cDGL  $(L, d)$ , and let  $f: \mathcal{L}_{[0,1]} \rightarrow (L, d)$  be a cDGL morphism, with  $f(a) = y$  and  $f(b) = z$ . A computation using (7) shows that the composition,

$$g: (\mathbb{L}(w), \partial) \xrightarrow{\Phi} \mathcal{L}_{[0,1]} \widehat{\otimes} \wedge (t, dt) \xrightarrow{f \otimes \operatorname{id}} (L, d) \widehat{\otimes} \wedge (t, dt),$$

verifies  $(\varepsilon_0 \circ g)(w) = y$  and  $(\varepsilon_1 \circ g)(w) = z$ , as required.

(2) Suppose now that  $y$  and  $z$  are right equivalent MC-elements in the cDGL,  $(L, d_L)$ , and let  $g: (\mathbb{L}(w), \partial) \rightarrow (L, d_L) \widehat{\otimes} \wedge (t, dt)$  be a cDGL morphism with  $\varepsilon_0 \circ g(w) = y$  and  $\varepsilon_1 \circ g(w) = z$ . Write  $g(w) = \alpha(t) - \beta(t)dt$ , with  $\alpha(t)$  and  $\beta(t)$  in  $L \widehat{\otimes} \wedge (t)$ . Since  $\partial w = -\frac{1}{2}[w, w]$ , the element  $\alpha(t)$  satisfies the differential equation,

$$\frac{d\alpha(t)}{dt} = d_L \beta(t) - [\beta(t), \alpha(t)].$$

By [6, Section 2], the solution of this equation, with initial data  $\alpha(0) = y$ , is

$$\alpha(t) = e^{-t \operatorname{ad}_{\beta(t)}}(y) + \frac{e^{-t \operatorname{ad}_{\beta(t)}} - 1}{-\operatorname{ad}_{\beta(t)}}(d_L \beta(t)).$$

We evaluate this expression in  $t = 1$  and we extract the value of  $d_L\beta(1)$  as

$$d_L\beta(1) = \text{ad}_{\beta(1)}(y) + \frac{\text{ad}_{-\beta(1)}}{e^{-\text{ad}_{\beta(1)}} - 1}(z - y).$$

From (2), we observe that this formula allows the construction of a cDGL map,  $f: \mathcal{L}_{[0,1]} \rightarrow (L, d_L)$ , defined by  $f(a) = y$ ,  $f(b) = z$  and  $f(x) = \beta(1)$ .  $\square$

We prove now that some particular quasi-isomorphisms preserve the set of equivalence classes of Maurer-Cartan elements. Let  $U$  be a graded vector space and let  $sU$  be the vector space isomorphic to  $U$  and whose gradation is given by  $|su| = |u| + 1$ . We form a cDGL,  $(\widehat{\mathbb{L}}(U \oplus sU), d)$ , with  $du = 0$  and  $dsu = u$ , for any  $u \in U$ .

**Proposition 2.4.** *For each cDGL,  $(A, d)$ , the injection,*

$$\iota: (A, d) \rightarrow (B, d) := (A, d) \coprod (\widehat{\mathbb{L}}(U \oplus sU), d),$$

*induces an isomorphism on the set of equivalence classes of Maurer-Cartan elements.*

*Proof.* First of all, since  $(A, d)$  is a retract of  $(B, d)$ , the map  $\widetilde{MC}(i)$  is injective. As for the surjectivity, let  $y$  be a MC-element in  $(B, d)$ . Denote by  $B(n)$  the vector space of brackets containing  $n$  times an element in  $U \oplus sU$ . Therefore, by definition, we have  $B(0) = A$  and we can write

$$y = \sum_{n=0}^{\infty} y_n,$$

where  $y_n \in B(n)$ . We observe that  $y_0$  is a MC-element in  $(A, d)$ , and that

$$dy_n = -\frac{1}{2} \sum_{p+q=n} [y_p, y_q]. \quad (8)$$

We now show that the elements  $y$  and  $y_0$  are right equivalent. For it, we introduce the derivation,  $s$ , of  $B$  of degree  $+1$  defined by  $s(A) = 0$ ,  $s(u) = su$  and  $s(su) = 0$ . The derivation  $\theta = sd + ds$  being equal to the identity on  $U \oplus sU$  is the multiplication by  $n$  on  $B(n)$ . If we set  $z_n = -s(y_{n+1})$ , we have

$$\begin{aligned} dz_n &= -ds(y_{n+1}) = -\theta(y_{n+1}) + sd(y_{n+1}) \\ &= -(n+1)y_{n+1} - \frac{1}{2}s \left( \sum_{p+q=n+1} [y_p, y_q] \right) \\ &= -(n+1)y_{n+1} - \sum_{p+q=n} [y_p, z_q]. \end{aligned} \quad (9)$$

With a direct calculation, we deduce from (8) and (9) that  $\alpha = \sum_{n \geq 0} y_n t^n + \sum_{n \geq 0} z_n t^n dt$  is a MC-element in  $(B, d) \widehat{\otimes} \wedge (t, dt)$ . Therefore, we can construct a cDGL map,  $g: (\mathbb{L}(w), \partial) \rightarrow (B, d) \widehat{\otimes} \wedge (t, dt)$ , by  $g(w) = \alpha$ . This map verifies  $(\varepsilon_0 \circ g)(w) = y_0$ ,  $(\varepsilon_1 \circ g)(w) = y$  and we have proved that  $y_0$  and  $y$  are right equivalent.  $\square$

We end this section with the following recall.

**Lemma 2.5.** [1, Lemma 4.6] *Let  $(\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), d)$  be a cDGL, where  $a$  is a Maurer-Cartan element and  $d(V) \subset \widehat{\mathbb{L}}(V)$ . Then, the injection  $(\widehat{\mathbb{L}}(V), d) \xrightarrow{\cong} (\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), d)$  is a quasi-isomorphism.*

### 3. MODELS OF SIMPLICIAL COMPLEXES

In [1], we have associated in a natural way a cDGL,  $\mathcal{L}_X$ , to each finite simplicial complex  $X$ . We recall here the main facts related to this construction. Denote by  $\Delta^n$  the simplicial complex whose set of  $p$ -simplices is

$$\Delta_p^n = \{(i_0, \dots, i_p) \mid 0 \leq i_0 < \dots < i_p \leq n\},$$

and by  $\wedge^n$  the subsimplicial complex whose  $p$ -simplices are sequences  $(i_0, \dots, i_p)$  that do not contain the sequence  $(0, \dots, n-1)$ . We denote by  $(C_*(\Delta^n), \delta)$  the usual simplicial chain complex on  $\Delta^n$ . The model  $\mathcal{L}_{\Delta^n}$  is constructed inductively and is characterized as follows.

**Proposition 3.1.** [1, Theorem 2.8] *The cDGL  $\mathcal{L}_{\Delta^n}$  is defined, up to isomorphism, by the following properties.*

- (i) *The cDGL's  $\mathcal{L}_{\Delta^n}$  are natural with respect to the injections of the subcomplexes  $\Delta^p$ , for all  $p < n$ .*
- (ii) *For  $n = 0$ , we have  $\mathcal{L}_{\Delta^0} = (\widehat{\mathbb{L}}(a), d)$  where  $a$  is a Maurer-Cartan element.*
- (iii) *The linear part  $d_1$  of the differential of  $\mathcal{L}_{\Delta^n}$  is the desuspension of the differential  $\delta$  of the chain complex  $C_*(\Delta^n)$ .*

For  $n = 1$ , this proposition implies that the cDGL  $\mathcal{L}_{\Delta^1}$  is the Lawrence-Sullivan interval  $\mathcal{L}_{[0,1]}$ , recalled in Section 1.

For each simplicial complex  $X$  contained in  $\Delta^n$ , by naturality, the Lie subalgebra  $\widehat{\mathbb{L}}(s^{-1}C_*(X))$  is preserved by the differential  $\partial$  of  $\mathcal{L}_{\Delta^n}$  and gives a model for  $X$ ,

$$\mathcal{L}_X = (\widehat{\mathbb{L}}(s^{-1}(C_*(X))), \partial).$$

In  $\mathcal{L}_{\Delta^n}$ , we denote by  $a_{i_0 \dots i_p}$  the generator of degree  $p-1$  corresponding to the simplex  $(i_0, \dots, i_p)$ . Each generator  $a_j$  corresponding to a vertex is a Maurer-Cartan element; we denote by  $\partial_{a_j} = \partial + \text{ad}_{a_j}$  the corresponding perturbed differential. An important feature of the construction is that, for any  $n \geq 2$ ,

$$\partial_{a_0}(a_{0 \dots n}) \in \mathcal{L}_{\partial \Delta^n},$$

where  $\partial \Delta^n$  denotes the boundary of  $\Delta^n$ . For this differential, the homologies of  $\mathcal{L}_{\Delta^n}$  and  $\mathcal{L}_{\wedge^n}$  are trivial, i.e.,

$$H(\mathcal{L}_{\Delta^n}, \partial_{a_0}) = H(\mathcal{L}_{\wedge^n}, \partial_{a_0}) = 0.$$

For  $n = 2$ , the differential  $\partial_{a_0}(a_{012})$  is given by a Baker-Campbell-Hausdorff product,

$$\partial_{a_0}(a_{012}) = a_{01} * a_{12} * a_{02}^{-1}.$$

For  $n \geq 3$ , the construction of  $\mathcal{L}_{\Delta^n}$  is done inductively. Suppose that  $\mathcal{L}_{\Delta^{n-1}}$  has been constructed and let  $x = \partial_{a_0}(a_{0 \dots n-1}) \in \mathcal{L}_{\partial \Delta^{n-1}} \subset \mathcal{L}_{\wedge^n}$ . Since  $H(\mathcal{L}_{\wedge^n}, \partial_{a_0}) = 0$ , there exists  $y \in \mathcal{L}_{\wedge^n}$  such that  $x = \partial_{a_0}(y)$ . We set  $\Omega_n = (-1)^n(a_{0 \dots n-1} - y)$  and  $\partial_{a_0}(a_{0 \dots n}) = \Omega_n$ . Therefore

$$\partial_{a_0} a_{0 \dots n} - (-1)^n a_{0 \dots n-1} \in \mathcal{L}_{\wedge^n}.$$



Theorem 1.2, corresponding to the case  $\Delta^1$ , can be generalized to any  $\Delta^n$  as follows.

**Proposition 3.2.** *There is a cDGL isomorphism,*

$$\varphi: (\mathcal{L}_{\wedge^n} \widehat{\Pi} \widehat{\mathbb{L}}(u, su), d) \rightarrow \mathcal{L}_{\Delta^n},$$

where  $|u| = n - 2$ ,  $|su| = n - 1$ ,  $du = 0$  and  $dsu = u$ .

*Proof.* For  $n = 1$ , this is Theorem 1.2. For  $n = 2$ , the morphism  $\varphi$  is defined by  $\varphi(su) = a_{012}$  and  $\varphi(u) = \partial(a_{012})$ . This is an isomorphism because  $a_{01} = \varphi(u * a_{02} * a_{12}^{-1} - [su, a_0])$ . For  $n \geq 3$ , the morphism  $\varphi$  is defined by  $\varphi(su) = a_{0\dots n}$  and  $\varphi(u) = \partial_{a_0}(a_{0\dots n})$ .  $\square$

The lifting Lemma 1.4 admits also a generalization.

**Proposition 3.3.** *Let  $p: (L_1, d) \rightarrow (L_2, d)$  be a surjective cDGL morphism. For any commutative diagram in solid line,*

$$\begin{array}{ccc} \mathcal{L}_{\wedge^n} & \xrightarrow{f} & (L_1, d) \\ \downarrow \iota & \nearrow h & \downarrow p \\ \mathcal{L}_{\Delta^n} & \xrightarrow{g} & (L_2, d), \end{array}$$

there exists a cDGL morphism,  $h$ , such that  $p \circ h = g$  and  $h \circ \iota = f$ .

*Proof.* For  $n = 1$  this is Lemma 1.4. In the general case, recall the isomorphism of Proposition 3.2,

$$\varphi: (\mathcal{L}_{\wedge^n} \widehat{\Pi} \widehat{\mathbb{L}}(u, su), d) \rightarrow \mathcal{L}_{\Delta^n},$$

with  $|u| = n - 2$ ,  $|su| = n - 1$ ,  $du = 0$ ,  $dsu = u$ .

Let  $x = g \circ \varphi(v) \in L_2$ . By surjectivity of  $p$ , there is an element  $y \in L_1$  with  $p(y) = x$ . We define  $h': (\mathcal{L}_{\wedge^n} \widehat{\Pi} \widehat{\mathbb{L}}(u, su), d) \rightarrow (L, d)$  by  $h'(su) = y$  and  $h'(u) = dy$ . Then the map  $h = h' \circ \varphi^{-1}$  is the required lifting.  $\square$

We recall and establish some useful lemmas. The first one is proved in [1] and the second one is a particular case of it.

**Lemma 3.4.** [1, Proposition 2.4] *Let  $J$  be the ideal generated by  $W$  in the cDGL,  $(\widehat{\mathbb{L}}(V \oplus W), d)$ . Suppose that  $d(J) \subset J$  and that, for the linear part  $d_1$  of  $d$ , we have  $H(W, d_1) = 0$ . Then, the projection  $(\widehat{\mathbb{L}}(V \oplus W), d) \rightarrow (\widehat{\mathbb{L}}(V), d)$  is a quasi-isomorphism.*

**Lemma 3.5.** *Let  $(\widehat{\mathbb{L}}(V), d)$  be a DGL. Suppose that  $a$  and  $b$  are two different Maurer-Cartan elements in  $V_{-1}$  and that  $x \in V_0$  is a Lawrence-Sullivan path from  $a$  to  $b$ . Then, the ideal  $I$  generated by  $b - a$  and  $x$  is acyclic.*

Note that any morphism of complete Lie algebras,  $f: \widehat{\mathbb{L}}(V) \rightarrow \widehat{\mathbb{L}}(W)$ , can be written as  $f = \sum_{i=1}^{\infty} f_i$  with  $f_i(V) \subset \mathbb{L}^i(W)$ .

**Lemma 3.6.** *If  $f_1$  is an isomorphism then  $f$  is an isomorphism.*

*Proof.* By induction on  $n$ , the map  $f$  induces isomorphisms,

$$f_{(n)}: \mathbb{L}(V)/\mathbb{L}^{>n}(V) \rightarrow \mathbb{L}(W)/\mathbb{L}^{>n}(W).$$

Therefore,  $f = \varprojlim_n f_{(n)}$  is also an isomorphism.  $\square$

*Remark 3.7.* In Lemma 3.6, the completeness of the Lie algebras is important. Consider, for instance, the morphism  $f: \mathbb{L}(a, b) \rightarrow \mathbb{L}(u, v)$ , defined by  $f(a) = u$  and  $f(b) = v + [u, v]$ ,  $|u| = |a| = 0$ ,  $|b| = |v| = n \in \mathbb{Z}$ . Then  $f_1$  is an isomorphism but not  $f$ . However, the map  $\hat{f}: \widehat{\mathbb{L}}(a, b) \rightarrow \widehat{\mathbb{L}}(u, v)$ , defined by  $f$ , is an isomorphism. In particular, we have

$$v = \hat{f} \left( \sum_{i=0}^{\infty} (-1)^i \text{ad}_a^i(b) \right).$$

When  $X$  is a finite simply connected simplicial complex, we have proved in [1, Theorem 7.4] that the model  $\mathcal{L}_X$  is quasi-isomorphic to the Quillen model of  $X$  introduced in [10]. As a first class of examples not covered by [10], we study models of Riemann surfaces. They are particular case of the next series of examples.

Let  $X$  be a space obtained by attaching a 2-cell along a wedge of circles. Such space can be seen as the union of a wedge of  $n$  circles with a polygon, in which each face is identified with one of the circles. Let  $x_1, \dots, x_n$  be generators of the fundamental group of the circles and let  $\omega = x_{i_1}^{\varepsilon_1} \dots x_{i_p}^{\varepsilon_p}$  be the word associated to the addition of the 2-cell. It is well known that the fundamental group of  $X$ , pointed at a vertex  $v_a$ , is given by the presentation

$$\pi_1(X) = \langle x_1, \dots, x_n : \omega \rangle.$$

The model of  $\mathcal{L}_X$  has a similar form.

**Proposition 3.8.** *Let  $a$  be a Maurer-Cartan element of  $\mathcal{L}_X$  corresponding to the vertex  $v_a$  of  $X$ . The cDGL  $(\mathcal{L}_X, \partial_a)$  is quasi-isomorphic to  $(\widehat{\mathbb{L}}(x_1, \dots, x_n, y), d)$  with  $|x_i| = 0$ ,  $|y| = 1$  and the differential  $d$  defined by  $dx_i = 0$ ,  $dy = x_{i_1}^{\varepsilon_1} * \dots * x_{i_p}^{\varepsilon_p}$ , where  $*$  denotes the Baker-Campbell-Hausdorff product.*

For instance, the model of the Klein Bottle,  $B$ , is given by

$$\mathcal{L}_B = (\widehat{\mathbb{L}}(u, v, x), d)$$

with  $du = dv = 0$ ,  $|u| = |v| = 0$ ,  $|x| = 1$  and

$$dx = u * v * u * (-v) = 2u + [v, u] + \dots$$

*Proof of Proposition 3.8.* A finite number of subdivisions of the circles and of the polygon gives a finite simplicial complex,  $Y$ , homeomorphic to  $X$ . The model  $\mathcal{L}_Y$  has generators in degree  $-1, 0$  and  $1$ , the edges with their boundary being Lawrence-Sullivan intervals and the triangles becoming generators of degree  $1$ . By using inductively Lemmas 3.9 and 3.5 on  $(\mathcal{L}_Y, d_a)$ , we reduce the set of generators to an element  $a$  of degree  $-1$ ,  $n$  elements of degree  $0$  and one element of degree  $1$ . This model has the form  $(\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), d_a)$ , with  $d_a a = -\frac{1}{2}[a, a]$  and  $d_a V \subset \widehat{\mathbb{L}}(V)$ . Thus, this model is quasi-isomorphic to the quotient cDGL,  $(\widehat{\mathbb{L}}(V), d)$ .  $\square$

**Lemma 3.9.** *Let  $(L, d) = (\widehat{\mathbb{L}}(W \oplus S), d)$  be a cDGL, with  $W$  concentrated in degrees  $-1, 0$  and  $1$ ,  $a \in W_{-1}$  being a Maurer-Cartan element and  $S = \mathbb{Q}u \oplus \mathbb{Q}v \oplus \mathbb{Q}t$ , with  $|u| = |v| = 1$ ,  $|x| = 0$ ,  $d_a u = \omega * x$ ,  $d_a v = x^{-1} * \omega'$ , where  $\omega, \omega' \in \widehat{\mathbb{L}}(W)$ . We suppose also that  $d_a(W) \subset \widehat{\mathbb{L}}(W)$ . Then  $(L, d_a)$  is quasi-isomorphic to  $(\widehat{\mathbb{L}}(W \oplus \mathbb{Q}t), d_a)$ , with  $d_a(t) = \omega * \omega'$ .*

*Proof.* Consider first the cDGL,

$$(\widehat{\mathbb{L}}(\alpha, \beta, v_1, v_2), d), |\alpha| = |\beta| = 0, d(\alpha) = d(\beta) = 0, |v_1| = |v_2| = 1, d(v_1) = \alpha, d(v_2) = \beta.$$

By replacing in each term of  $\alpha * \beta$  one, and only one,  $\alpha$  or  $\beta$  by  $v_1$  or  $v_2$  respectively, we get an element  $\gamma$  with  $d(\gamma) = \alpha * \beta$ . Consider now the morphism,

$$\psi: (\widehat{\mathbb{L}}(\alpha, \beta, v_1, v_2), d) \rightarrow (L, d),$$

defined by  $\psi(\alpha) = \omega * x$ ,  $\psi(\beta) = x^{-1} * \omega'$ ,  $\psi(v_1) = u$  and  $\psi(v_2) = v$ . If we denote  $z = \psi(\gamma)$ , then  $dz = \omega * x * x^{-1} * \omega' = \omega * \omega'$ . Moreover, the linear part of  $z$  is  $u + v$ . The morphism  $\psi$  induces an isomorphism

$$\Psi: (L', d') := (\widehat{\mathbb{L}}(W \oplus \mathbb{Q}r \oplus \mathbb{Q}s \oplus \mathbb{Q}t), d_a) \rightarrow (L, d).$$

Here  $d_a(r) = 0$ ,  $|r| = 0$ ,  $d_a(s) = r$ ,  $|s| = 1$ ,  $d_a(t) = \omega * \omega'$ ,  $|t| = 1$ ,  $\Psi(r) = \omega * x$ ,  $\Psi(s) = u$ ,  $\Psi(t) = z$ . Finally, using Lemma 3.4, we deduce that  $(L', d')$  is quasi-isomorphic to  $(\widehat{\mathbb{L}}(W \oplus \mathbb{Q}t), d_a)$ .  $\square$

#### 4. WEAK-EQUIVALENCES

In this section, we introduce the weak-equivalences of our closed model structure. Define them as quasi-isomorphisms should identify the models  $\mathcal{L}_X$ ,  $\mathcal{L}_Y$  of any finite simplicial complexes,  $X$  and  $Y$ , as shows the following result.

**Theorem 4.1.** *Let  $X$  be a connected finite simplicial complex, then  $H(\mathcal{L}_X) = 0$ .*

*Proof.* Write  $\mathcal{L}_X = (\widehat{\mathbb{L}}(V), d)$ . We choose an element  $a \in V_{-1}$  corresponding to a vertex. We denote by  $W$  the subspace of  $V$  defined by  $W_n = V_n$  for  $n \geq 0$  and such that  $W_{-1}$  is the subspace of  $V_{-1}$  generated by the differences  $b_i - a$  where the  $b_i$ 's browse the vertices of  $X$ . Then, we have an isomorphism of complete Lie algebras,

$$\widehat{\mathbb{L}}(V) \cong \mathbb{L}(a) \widehat{\Pi} \widehat{\mathbb{L}}(W).$$

In  $\mathcal{L}_X$ , the differential  $d$  satisfies

$$da = -\frac{1}{2}[a, a] \quad \text{and} \quad dx = d_a x - [a, x].$$

Denote by  $\mathcal{I}$  the ideal generated by  $W$  in  $\mathcal{L}_X$ . Since  $V$  is finite dimensional and  $|a| = -1$ , the ideal  $\mathcal{I}$  is the complete free Lie algebra on the vector space  $\overline{W}$ , of basis  $(\text{ad}_a^n(v_i))_{i,n}$  where  $(v_i)_i$  is a basis of  $W$  and  $n \geq 0$ , i.e.,

$$\mathcal{I} = \widehat{\mathbb{L}}(\overline{W}).$$

We denote by  $d_1$  the linear part of the differential  $d$  in  $\mathcal{L}_X$  and by  $\delta$  the linear part of the differential induced by  $d$  in  $\mathcal{I} = \widehat{\mathbb{L}}(\overline{W})$ . An induction shows that

$$\delta(\text{ad}_a^n(v)) = \begin{cases} (-1)^n \text{ad}_a^n(d_1(v)) - \text{ad}_a^{n+1}(v) & \text{if } n \text{ is odd,} \\ (-1)^n \text{ad}_a^n(d_1(v)) & \text{if } n \text{ is even.} \end{cases}$$

From this computation, we deduce that if  $\sum_{n,i} \alpha_{n,i} \text{ad}_a^n(v_i)$ ,  $\alpha_{n,i} \in \mathbb{Q}$ , is a  $\delta$ -cycle, then we have  $\alpha_{2n+1,i} = 0$ , for any  $n$  and  $i$ , and  $\sum_{n,i} \alpha_{2n,i} \text{ad}_a^{2n}(d_1 v_i) = 0$ . Thus, we obtain

$$\sum_{n,i} \alpha_{n,i} \text{ad}_a^n(v_i) = -\delta\left(\sum_{n,i} \alpha_{2n,i} \text{ad}_a^{2n-1}(v_i)\right).$$

Therefore,  $H(\mathcal{J}, \delta) = 0$  and from Lemma 3.4, we deduce that  $\mathcal{J}$  is  $d$ -acyclic. Finally, since  $\mathcal{L}_X/\mathcal{J} = (\mathbb{L}(a), d)$ , we get  $H(\mathcal{L}_X) = 0$ .  $\square$

**Definition 4.2.** A cDGL morphism,  $f: (L', d) \rightarrow (L, d)$ , is a *weak-equivalence* if it induces a bijection between the classes of Maurer-Cartan elements and a quasi-isomorphism,  $(L, d_a) \rightarrow (L', d_{f(a)})$ , for any perturbed differential  $d_a = d + \text{ad}_a$  where  $a$  browses the set of Maurer-Cartan elements.

The next result gives examples of weak-equivalences.

**Proposition 4.3.** *For each cDGL,  $(A, d)$ , the injection,*

$$\iota: (A, d) \rightarrow (B, d) := (A, d) \coprod^{\widehat{\mathbb{L}}(U \oplus sU)} (\widehat{\mathbb{L}}(U \oplus sU), d), \text{ with } d: sU \cong U,$$

*is a weak-equivalence.*

*Proof.* The fact that  $\iota$  induces a bijection between the classes of MC-elements is established in Proposition 2.4. Therefore, the proof is reduced to the fact that  $\iota$  induces an isomorphism in homology, for any perturbed differential.

Let  $a \in A$  be a MC-element. The differential perturbed by  $a$  verifies  $d_a(su) = u + [a, su]$ ,  $d_a(u) = [a, u]$ . We consider now the spectral sequence associated to the filtration by the number of elements belonging to  $A$  in a bracket. The  $\delta^0$ -differential of this spectral sequence verifies  $\delta^0(su) = u$ ,  $\delta^0(u) = 0$ . Thus, the abutment is the homology  $H(A, d_a)$ .  $\square$

We come back to the lifting problem of paths (Lemma 1.4) with a surjective weak-equivalence,  $p$ . Before stating the result, recall from (2), that the perturbed differential of the Lawrence-Sullivan interval verifies,

$$\partial_a(x) = \frac{\text{ad}_{-x}}{e^{-\text{ad}_x} - 1}(b - a).$$

Therefore, if  $(L, d)$  is a cDGL,  $v$  a Maurer-Cartan element in  $L$  and  $c$  a  $d_a$ -cycle of degree zero, then there exists a cDGL morphism,  $f_c: (\mathcal{L}_{[0,1]}, \partial_a) \rightarrow (L, d_v)$ , defined by  $f_c(a) = f_c(b) = v$  and  $f_c(x) = c$ .

In the next result and also in Proposition 5.4, we use the following basic fact.

**Lemma 4.4.** *Let  $p: (L', d) \rightarrow (L, d)$  be a surjective quasi-isomorphism. Then the following properties are satisfied.*

- 1) *For any cycle  $z \in L$ , there exists a cycle  $z' \in L'$  such that  $p(z') = z$ .*
- 2) *For any cycle  $z' \in L'$  such that  $p(z') = dx$ , there exists  $x' \in L'$  such that  $z' = dx'$  and  $p(x') = x$ .*

**Proposition 4.5.** *Let  $p: (L_1, d) \rightarrow (L_2, d)$  be a surjective weak-equivalence of cDGL's. Suppose that  $u$  and  $v$  are Maurer-Cartan elements in  $L_1$ , such that  $p(u)$  and  $p(v)$  are connected by a Lawrence-Sullivan interval,  $f: \mathcal{L}_{[0,1]} \rightarrow (L_2, d)$ . Then, there is a cDGL morphism,  $h: \mathcal{L}_{[0,1]} \rightarrow (L_1, d)$ , with  $h(a) = u$ ,  $h(b) = v$  and  $p \circ h = f$ ,*

$$\begin{array}{ccc} & & (L_1, d) \\ & \nearrow h & \downarrow p \\ \mathcal{L}_{[0,1]} & \xrightarrow{f} & (L_2, d). \end{array}$$

*Proof.* By Lemma 1.4, there is a morphism  $h': \mathcal{L}_{[0,1]} \rightarrow (L_1, d)$  with  $p \circ h' = f$  and  $h'(a) = u$ . Since  $\widetilde{MC}(f)$  is a bijection, there is an interval  $h'': \mathcal{L}_{[0,1]} \rightarrow (L_1, d)$  with  $h''(a) = h'(b)$  and  $h''(b) = v$ . By construction, the projection  $p \circ h''(x)$  is a  $d_{p(v)}$ -cycle and, using Lemma 4.4, there is a  $d_v$ -cycle,  $c \in L_1$ , such that  $p(c) = -p \circ h''(x)$ . The cycle  $c$  induces a map,

$$f_c: \mathcal{L}_{[0,1]} \rightarrow (L_1, d),$$

by  $f_c(a) = f_c(b) = v$  and  $f_c(x) = c$ . The composition of the three intervals  $h'$ ,  $h''$  and  $f_c$  gives the lifting.  $\square$

## 5. THE STRUCTURE OF CLOSED MODEL CATEGORY

Denote by **cDGL** the category of complete differential graded Lie algebras  $(L, d)$  such that  $H_*(L, d) = H_{\geq 0}(L, d)$ .

**Theorem 5.1.** *Define a map,  $f: (L', d) \rightarrow (L, d)$  in **cDGL** to be*

- *a weak-equivalence if it satisfies to the requirements of Definition 4.2,*
- *a fibration if it is a surjection,*
- *a cofibration if it has the LLP with respect to all trivial fibrations.*

*Then, **cDGL** is a closed model category.*

We prove this result by decomposing it in several steps. First, we define a subclass of cofibrations.

**Definition 5.2.** *A free map is a map  $f: (L', d) \rightarrow (L, d)$  such that:*

- (1) *as complete Lie algebra,  $L$  is isomorphic to  $L' \coprod \widehat{\mathbb{L}}(W)$  and  $f$  is isomorphic to the canonical inclusion;*
- (2)  *$W = U \oplus sU \oplus V$  with  $dsu = u$ ,  $du = 0$ ;*
- (3)  *$V = V_{\geq -1}$  and  $V_{-1}$  is generated by Maurer-Cartan elements;*
- (4)  *$V_0 = V'_0 \oplus V''_0$  where each element of  $V'_0$  is a cycle and for each element  $x \in V''_0$  there exist Maurer-Cartan elements,  $a$  and  $b$ , in  $L_{-1}$  such that  $\widehat{\mathbb{L}}(a, b, x)$  is a sub cDGL, isomorphic to the Lawrence Sullivan algebra  $\mathcal{L}_{[0,1]}$ ;*
- (5) *for any  $x \in V_n$ , with  $n \geq 1$ , there exists a Maurer-Cartan element,  $a$ , such that  $d_a V_n \subset L' \coprod \widehat{\mathbb{L}}(V_{<n})$ .*

**Definition 5.3.** *A complete cDGL,  $(L, d)$ , is called *inductive* if  $0 \rightarrow (L, d)$  is a free map.*

For instance, the cDGL  $\mathcal{L}_X$  associated to a finite simplicial complex  $X$  is an inductive cDGL. Also, any cDGL of the form  $(\widehat{\mathbb{L}}(W), d)$  with  $W = W_{\geq 0}$  is inductive.

**Proposition 5.4.** *Any free map is a cofibration.*

*Proof.* Let  $f: (L', d) \rightarrow (L, d) = (L' \coprod \widehat{\mathbb{L}}(U \oplus sU \oplus V), d)$  be a free map. We consider the following commutative diagram,  $p \circ \psi = \varphi \circ f$ , where  $p$  is a trivial fibration.

$$\begin{array}{ccc} (L', d) & \xrightarrow{\psi} & (A, d) \\ f \downarrow & \nearrow \Phi & \downarrow p \\ (L, d) & \xrightarrow{\varphi} & (B, d) \end{array}$$

We have to construct  $\Phi$ . First, as  $p$  is surjective, if  $su \in sU$ , there exists  $x \in A$  such that  $p(a) = \varphi(su)$ . We set  $\Phi(su) = a$  and  $\Phi(u) = da$ . We are now reduced to the particular case where  $(L, d) = (L' \hat{\coprod} \hat{\mathbb{L}}(V), d)$ . We choose a basis  $(a_i)_{i \in I}$  of  $V$  satisfying the following properties.

- (i) The set  $I$  is an ordered set such that if  $|a_i| < |a_j|$ , then  $i < j$ .
- (ii) The elements  $a_i$  of degree  $-1$  are Maurer-Cartan elements.
- (iii) For each element  $a_i$  of degree  $0$ , either  $da_i = 0$ , or else there are elements  $a_{i_1}$  and  $a_{i_2}$  of degree  $-1$  in  $V$  such that  $(\hat{\mathbb{L}}(a_i, a_{i_1}, a_{i_2}), d)$  is isomorphic to the LS-interval,  $\mathcal{L}_{[0,1]}$ .

• If  $|a_i| = -1$ , then, since  $p$  is a surjective weak-equivalence, there is a Maurer-Cartan element  $u$  in  $A$  with  $p(u) = \varphi(a_i)$ . We extend  $\Phi$  by  $\Phi(a_i) = u$ .

• If  $|a_i| = 0$  and  $da_i \neq 0$ , we denote by  $(\hat{\mathbb{L}}(a_{i_0}, a_{i_1}, a_i), d)$  the Lawrence-Sullivan interval associated to  $a_i$ . By induction,  $\Phi$  is defined on the sub cDGL  $(\hat{\mathbb{L}}(a_{i_0}, a_{i_1}), d)$ . The result follows from Proposition 4.5.

• Now suppose  $\Phi(x_j)$  defined when  $|x_j| < n$  with  $n \geq 1$ . Let  $x$  be a generator with  $|x| = n$ . We denote by  $a$  the associated Maurer-Cartan element to  $x$ , as required in the definition of free map, and form the next commutative diagram,

$$\begin{array}{ccc} & (A, d_{\Phi(a)}) & \\ & \nearrow \Phi & \downarrow p \\ (L, d_a) & \xrightarrow{\varphi} & (B, d_{\varphi(a)}), \end{array}$$

where  $\Phi$  is already defined on  $L' \hat{\coprod} \hat{\mathbb{L}}(V_{<n})$ . The element  $\Phi d_a(x)$  is a cycle in  $A$  with  $p(\Phi d_a(x)) = d_{\varphi(a)} \varphi x$ . Therefore by Lemma 4.4, there exists  $y \in A$  with  $\Phi d_a(x) = d_{\Phi(a)} y$  and  $p(y) = \varphi(x)$ . We extend  $\Phi$  by putting  $\Phi(x) = y$ . We now observe that  $\Phi(dx) = \Phi(d_a x) - \Phi([a, x]) = d_{\Phi(a)} y - [\Phi(a), y] = dy = d\Phi(x)$ .  $\square$

**Proposition 5.5.** *Any map,  $f: (L', d) \rightarrow (L, d)$ , in **cDGL** admits a factorization  $f = p \circ \iota$  where  $p$  is a fibration and  $\iota$  a trivial cofibration.*

*Proof.* We set  $W = L \oplus s^{-1}L$  and  $dl = s^{-1}l$ ,  $ds^{-1}l = 0$ , for any  $l \in L$ . We define  $p: \hat{\mathbb{L}}(W) \rightarrow L$  by  $p(l) = l$  and  $p(s^{-1}l) = dl$ . We have factorized  $f$  in a fibration and a free map,

$$L' \twoheadrightarrow L' \hat{\coprod} \hat{\mathbb{L}}(W) \xrightarrow{p} L.$$

Proposition 5.4 implies that  $\iota$  is a cofibration and Proposition 2.4 that it is also a weak-equivalence.  $\square$

**Proposition 5.6.** *Any map,  $f: (L', d) \rightarrow (L, d)$ , in **cDGL** admits a factorization  $f = p \circ \iota$  where  $p$  is a trivial fibration and  $\iota$  a cofibration.*

*Proof.* We begin with the factorization of Proposition 5.5,

$$(L', d) \twoheadrightarrow (L' \hat{\coprod} \hat{\mathbb{L}}(W), d) \xrightarrow{p} (L, d).$$

What we have to do is to add elements to  $W$ , with the aim of transforming  $p$  in a weak-equivalence. In a first step, we add a vector space,  $V(0)$ , built on Maurer-Cartan

elements and cycles in such a way that

$$p_0: (L'', d) = (L' \coprod \widehat{\mathbb{L}}(W \oplus V(0)), d) \rightarrow (L, d)$$

is surjective on the classes of Maurer Cartan elements and induces surjections in homology for each differential perturbed by a Maurer-Cartan element. To get the injections, we consider two steps.

- Step 1. If two MC-elements,  $a$  and  $b$ , in  $(L'', d)$  are sent to the same class in  $(L, d)$ , we add an element  $x$  of degree 0 with a differential such that  $\mathbb{L}(a, b, x)$  is a LS-interval.
- Step 2. For any MC-element,  $a$ , and any non trivial  $d_a$ -cycle,  $x$ , such that  $p(x) = d_{p(a)}(u)$ , we add a new generator  $y$  with  $d_a(y) = x$  and  $p(y) = u$ .

We denote by  $V(1)$  the new generators created by these two steps. Let  $\omega_1$  be the first non-numerable ordinal. We repeat the two previous steps, again and again, to obtain

$$p_{\omega_1}: (M, d) := (L' \coprod \widehat{\mathbb{L}}(W \oplus (\oplus_{i \leq \omega_1} V(i))), d) \rightarrow (L, d).$$

Denote  $W_{\omega_1} = W \oplus (\oplus_{i \leq \omega_1} V(i))$ . Observe that the elements of  $W_{\omega_1}$  respect the rules imposed in Definition 5.2. Therefore, the injection  $\iota: (L, d) \hookrightarrow (M, d)$  is a free map. We claim that the composition  $p_{\omega_1} \circ \iota$  is the expected factorization of  $f$ . To prove that, we have to show that  $p_{\omega_1}$  induces an injection,  $H(p_{\omega_1}): H(M, d_a) \rightarrow H(L, d_{p_{\omega_1}(a)})$ , for any MC-element  $a$  of  $(M, d)$  and an injection between the classes of MC-elements.

Let  $a$  be a MC-element in  $(M, d)$  and  $\alpha$  be a  $d_a$ -cycle in  $M$  such that there exists  $x \in L$  with  $p(\alpha) = d_{p(a)}(x)$ . The cycle  $\alpha$  can be written as a series involving brackets of elements of  $M$  and of  $W_{\omega_1}$ . Thus, there exists  $j < \omega_1$  with  $\alpha \in L' \coprod \widehat{\mathbb{L}}(W \oplus (\oplus_{i \leq j} V(i)))$ . By construction, there exists  $\beta \in L' \coprod \widehat{\mathbb{L}}(W \oplus (\oplus_{i \leq j+1} V(i)))$  such that  $d\beta = \alpha$ . We have established the injectivity of  $H(p_{\omega_1})$ . The proof for the injectivity between classes of MC-elements is similar.  $\square$

*Proof of Theorem 5.1.* The axioms CM 1, 2, 3 of a closed model category structure [9] are obvious. The axiom CM 4 (i) is satisfied by definition of cofibrations. Proposition 5.5 is CM 5 (ii) and Proposition 5.6 is CM 5 (i).

For CM 4 (ii), we consider a trivial cofibration,  $f: (L', d) \rightarrow (L, d)$ , and decompose it as in Proposition 5.5:

$$(L', d) \xrightarrow{\iota} (L', d) \coprod \widehat{\mathbb{L}}(\widehat{\mathbb{L}}(s^{-1}L, L), d) \xrightarrow{p} (L, d).$$

It is easy to see (cf. the beginning of the proof of Proposition 5.4) that  $\iota$  has the LLP respect to surjective maps. Moreover, as  $f$  is a trivial cofibration,  $p$  is a trivial fibration. Thus  $f$  admits a lifting with respect to  $p$  and  $f$  is a retract of  $i$ . Therefore,  $f$  has also the LLP respect to surjective maps and CM 4 (ii) is established.  $\square$

## 6. THE CYLINDER CONSTRUCTION

First, we recall the construction of a cylinder for cDGL's, made in [13].

Let  $(L, d)$  be a cDGL of the form  $L = \widehat{\mathbb{L}}(V)$ . We denote by  $V'$  a copy of  $V$ , the isomorphism  $V \xrightarrow{\cong} V'$  being represented by  $v \mapsto v'$ . We construct a cDGL,

$$(\widehat{\mathbb{L}}(V \oplus V' \oplus sV'), d),$$



where  $d|_V$  is the differential on  $(L, d)$  and, if  $v' \in V'$ , we have  $dv' = 0$  and  $dsv' = v'$ .

A derivation  $i$  of degree  $+1$  is defined on  $(\widehat{\mathbb{L}}(V \oplus V' \oplus sV'), d)$  by  $i(v) = sv'$ ,  $i(v') = i(sv') = 0$ . Then  $\theta = i \circ d + d \circ i$  is a derivation commuting with  $d$ . Therefore  $e^\theta$  is an automorphism. We choose new copies  $\widehat{V} \cong V$  and  $\overline{V} \cong sV$  and define a cDGL morphism,

$$\psi: \widehat{\mathbb{L}}(V \oplus \widehat{V} \oplus \overline{V}) \rightarrow \widehat{\mathbb{L}}(V \oplus V' \oplus sV'), \quad (10)$$

by  $\psi(v) = v$ ,  $\psi(\widehat{v}) = e^\theta(v)$  and  $\psi(\overline{v}) = sv$ . This is an isomorphism and induces a differential  $D = \psi^{-1} \circ d \circ \psi$  on  $\widehat{\mathbb{L}}(V \oplus \widehat{V} \oplus \overline{V})$ . We denote by  $\text{Cyl}(L, d)$  this cDGL.

**Definition 6.1.** Let  $(L, d)$  be a cDGL with  $L \cong \widehat{\mathbb{L}}(V)$ . The *cylinder construction* on  $(L, d)$  is the cDGL,  $\text{Cyl}(L, d) = (\widehat{\mathbb{L}}(V \oplus \widehat{V} \oplus \overline{V}), D)$ , together with the maps

$$(L, d) \xrightleftharpoons[\lambda_1]{\lambda_0} \text{Cyl}(L, d) \xrightarrow{p} (L, d),$$

defined by  $\lambda_0(v) = v$ ,  $\lambda_1(v) = e^\theta(v)$ ,  $p(v) = v$ ,  $p(\widehat{v}) = p(\overline{v}) = 0$ .

*Remark 6.2.* Since  $e^\theta$  is an automorphism commuting with the differential, the sub Lie algebra  $\widehat{\mathbb{L}}(\widehat{V})$  is a sub cDGL, isomorphic to  $(L, d)$ . The previous cylinder can be represented in two ways, up to isomorphism,

$$\begin{array}{ccc} (\widehat{\mathbb{L}}(V \oplus \widehat{V} \oplus \overline{V}), D) & \xrightarrow{\psi} & (\widehat{\mathbb{L}}(V \oplus V' \oplus sV'), d) \\ \swarrow \scriptstyle \iota_1 & & \nearrow \scriptstyle \lambda_1 \\ & (\widehat{\mathbb{L}}(V), d) & \\ \nwarrow \scriptstyle \iota_0 & & \nearrow \scriptstyle \lambda_0 \end{array}$$

where  $\iota_0: V \xrightarrow{\cong} V$  and  $\iota_1: V \xrightarrow{\cong} \widehat{V}$ .

The cylinder construction applied to the cDGL,  $(L, d) = (\mathbb{L}(a), \partial_a)$ ,  $\partial_a a = -\frac{1}{2}[a, a]$ , gives the Lawrence Sullivan construction, cf. Theorem 1.2. The iteration of the cylinder construction gives cubes of higher dimensions,  $\mathcal{M}_{I^1} = \mathcal{L}_{[0,1]}$  and  $\mathcal{M}_{I^n} = \text{Cyl}(\mathcal{M}_{I^{n-1}})$ . Moreover, from Lemma 3.4, we deduce  $H(\mathcal{M}_{I^n}, d_a) = 0$  for any  $n \geq 1$ . We prove now the main result concerning this construction.

**Theorem 6.3.** *Let  $(L, d)$  be an inductive cDGL with  $L \cong \widehat{\mathbb{L}}(V)$ . The cylinder construction  $\text{Cyl}(L, d)$  is a cylinder object in the closed model category **cDGL**.*

The main point in this theorem concerns the existence of cofibrations that we state independently.

**Proposition 6.4.** *If  $(L, d)$  is an inductive cDGL, the inclusions,  $\iota_0, \iota_1: (L, d) \rightarrow \text{Cyl}(L, d)$ , are free maps and, therefore, cofibrations.*

*Proof.* Let  $(L, d) = (\widehat{\mathbb{L}}(V), d)$  and  $\text{Cyl}(L, d) = (\widehat{\mathbb{L}}(V \oplus \widehat{V} \oplus \overline{V}), D)$ . We have to verify the requirements of Definition 5.2.

- For the elements of degree  $-1$ , this is clear by construction.
- For the elements of degree  $0$ , if they belong to  $V$  or to  $\widehat{V}$ , this comes from the fact that  $(L, d)$  is inductive. If the element of degree  $0$  belongs to  $\overline{V}$ , this is a consequence of Lemma 6.5.



- For the elements of degree  $n > 0$ , this follows from the inductive properties of  $(L, d)$  and from Lemma 6.6.  $\square$

For completing the previous proof, we describe in more detail the model of the cylinder of the Lawrence-Sullivan interval,

$$\text{Cyl}(\mathcal{L}_{[0,1]}) = (\widehat{\mathbb{L}}(a, b, x, \hat{a}, \hat{b}, \hat{x}, \bar{a}, \bar{b}, \bar{x}), D).$$

Here the sub cDGL's,  $(\widehat{\mathbb{L}}(a, b, x), \partial)$ ,  $(\widehat{\mathbb{L}}(\hat{a}, \hat{b}, \hat{x}), \partial)$ ,  $(\widehat{\mathbb{L}}(a, \hat{a}, \bar{a}), \partial)$  and  $(\widehat{\mathbb{L}}(b, \hat{b}, \bar{b}), \partial)$  are Lawrence-Sullivan intervals.

**Lemma 6.5.** *Let  $\mathcal{L}_{\partial I^2}$  be the sub cDGL generated by  $a, b, x, \hat{a}, \hat{b}, \hat{x}, \bar{a}$  and  $\bar{b}$ . There is an element  $\Omega \in \mathcal{L}_{\partial I^2}$  and an isomorphism,*

$$g: (\mathcal{L}_{\partial I^2} \widehat{\Pi} \mathbb{L}(u), d) \rightarrow \text{Cyl}(\mathcal{L}_{[0,1]}),$$

with  $|\Omega| = 0$ ,  $|u| = 1$ ,  $du = \Omega$ .

*Proof.* Let  $\mathcal{J}$  be the differential ideal generated by  $x, \hat{x}, \bar{a}, b - a, \hat{b} - \hat{a}$  and  $\hat{a} - a$ . Recall that the map  $\psi$  defined in (10) verifies  $\psi(D\bar{b}) = d\psi(\bar{b}) = dsb' = b'$  and, from (5), that  $b' = \psi(\partial x)$ . We deduce  $D\bar{b} = \partial x$  and

$$D_a \bar{b} = \partial_a x - [a, x] = \frac{\text{ad}_{-x}}{e^{-\text{ad}_x} - 1}(b - a) - [a, x].$$

In the quotient by  $\mathcal{J}$  this expression is equal to 0. Therefore, we have a cDGL map,  $\mathcal{L}_{\partial I^2} \rightarrow (\widehat{\mathbb{L}}(\alpha, \bar{b}), d_\alpha)$ , where  $\alpha$  is a MC-element corresponding to the class  $\{a, \hat{a}, b, \hat{b}\}$  and  $d_\alpha \bar{b} = 0$ . By Lemma 3.4, this map is a quasi-isomorphism and, by Lemma 2.5, the injection  $(\mathbb{L}(\bar{b}), 0) \hookrightarrow (\widehat{\mathbb{L}}(\alpha, \bar{b}), d_\alpha)$  is a quasi-isomorphism. We deduce the existence of an element  $\Omega \in \mathcal{L}_{\partial I^2}$ , of degree 0, such that  $\Omega = \bar{b} + \gamma$ , with  $\gamma \in \mathcal{J}$ , and

$$H(\mathcal{L}_{\partial I^2}, \partial_a) = \mathbb{Q} \Omega.$$

In  $\text{Cyl}(\mathcal{L}_{[0,1]})$ , there exists  $z$  such that  $\Omega = Dz$ . Thus, we may extend the inclusion in a cDGL map,

$$g: (\mathcal{L}_{\partial I^2} \widehat{\Pi} \mathbb{L}(u), d) \rightarrow \text{Cyl}(\mathcal{L}_{[0,1]}),$$

by  $du = \Omega$  and  $g(u) = z$ . As the linear part of  $D\bar{x}$  is  $D_1 \bar{x} = \bar{a} + \hat{x} - \bar{b} - x$ , we have  $z = -\bar{x} + \alpha$  with  $\alpha \in \mathcal{J}$ . Therefore  $g$  is an isomorphism on the indecomposable elements and, by Lemma 3.2, this is an isomorphism.  $\square$

For the next lemma, we choose a basis  $(a_i)_{i \in I}$  of  $V$ . We denote by  $(\hat{a}_i)_{i \in I}$ ,  $(\bar{a}_i)_{i \in I}$ ,  $(a'_i)_{i \in I}$  the corresponding basis of  $\widehat{V}$ ,  $\bar{V}$ ,  $V'$ .

**Lemma 6.6.** *Suppose  $(\widehat{\mathbb{L}}(V), d)$  inductive and let  $\text{Cyl}(\widehat{\mathbb{L}}(V), d) = (\widehat{\mathbb{L}}(V \oplus \widehat{V} \oplus \bar{V}), D)$ . If  $|a_i| = n \geq 1$ , the element  $D\bar{a}_i - (\hat{a}_i - a_i)$  belongs to the ideal  $\mathcal{J}$  generated by the elements  $a_j$ ,  $\bar{a}_j$  and  $\hat{a}_j$  with  $|a_j| < n$ .*

*Proof.* Since  $da_i \in \widehat{\mathbb{L}}(a_j; |a_j| < n)$ , we have

$$\psi(\hat{a}_i) = e^\theta(a_i) = a_i + b_i + \sum_{n \geq 1} \frac{(i \circ d)^n}{n!}(a_i) = \psi(a_i) + b_i + \psi(\gamma),$$

with  $\gamma \in \mathcal{J}$ . Therefore  $\psi(D\bar{a}_i) = d\psi(\bar{a}_i) = dc_i = b_i = \psi(\hat{a}_i - a_i - \gamma)$ , and  $D\bar{a}_i = \hat{a}_i - a_i - \gamma$ .  $\square$

*Proof of Theorem 6.3.* Observe first that  $p \circ \lambda_0 = p \circ \lambda_1 = \text{id}$ . Also, with Proposition 5.4, we know that the two maps,  $\lambda_0, \lambda_1$ , are cofibrations. As Proposition 4.3 implies that  $\lambda_0$  is a weak-equivalence, from the equality  $p \circ \lambda_0 = \text{id}$ , we deduce that  $p$  is a weak-equivalence also.  $\square$

Classically in a closed model category, we get a notion of homotopy from the existence of a cylinder object.

**Definition 6.7.** Let  $(\widehat{\mathbb{L}}(V), d)$  be an inductive cDGL. Two cDGL maps,  $f_0, f_1: (\widehat{\mathbb{L}}(V), d) \rightarrow (L', d)$  are *left homotopic* if there exists a cDGL map,  $F: \text{Cyl}(\widehat{\mathbb{L}}(V), d) \rightarrow (L', d)$ , such that  $f_0 = F \circ \lambda_0$  and  $f_1 = F \circ \lambda_1$ .

*Remark 6.8.* One can prove also that the sequence,

$$(L, d) \hookrightarrow (L, d)^I := (L, d) \hat{\otimes} \wedge (t, dt) \xrightarrow{(\varepsilon_0, \varepsilon_1)} (L, d) \hat{\otimes} (L, d),$$

is a path object. With this notion, the maps  $f_0, f_1: (\widehat{\mathbb{L}}(V), d) \rightarrow (L', d)$  are said *right homotopic* if there exists a cDGL map,  $G: (L, d) \rightarrow (L', d)^I$ , such that  $\varepsilon_0 \circ G = f_0$  and  $\varepsilon_1 \circ G = f_1$ . It is well known that both definitions of left and right homotopy for  $f_0, f_1$  coincide if the domain is cofibrant; we call it simply “*homotopy*”.

We introduce now a *cone construction* which gives another inductive process for building the models of  $\Delta^n$ .

**Definition 6.9.** If  $X$  is a simplicial complex, the projection to a point  $X \rightarrow \{*\}$  induces a morphism  $\mathcal{L}_X \rightarrow (\mathbb{L}(a), \partial)$ , where  $a$  is a Maurer-Cartan element. The cone on  $\mathcal{L}_X$  is the pushout

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{\rho} & (\mathbb{L}(a), \partial) \\ \downarrow \iota_1 & & \downarrow \\ \text{Cyl}(\mathcal{L}_X) & \longrightarrow & \text{Cone}(\mathcal{L}_X). \end{array}$$

**Theorem 6.10.** The model of  $\Delta^n$  is isomorphic to the cone on the model of  $\Delta^{n-1}$ ,

$$\mathcal{L}_{\Delta^n} \cong \text{Cone}(\mathcal{L}_{\Delta^{n-1}}).$$

*Proof.* By definition,  $\mathcal{L}_{\Delta^{n-1}} = (\widehat{\mathbb{L}}(\oplus a_{i_0 \dots i_p} \mathbb{Q}), \partial)$ , with  $0 \leq i_0 < \dots < i_p \leq n-1$  and the linear part of the differential is given by

$$\partial_1(a_{i_0 \dots i_p}) = \sum_{j=0}^p (-1)^j a_{i_0 \dots \hat{i}_j \dots i_p}.$$

Recall  $\iota_1(a_{i_0 \dots i_p}) = \hat{a}_{i_0 \dots i_p}$ . To construct the cone, we consider the projection  $\rho: \mathcal{L}_{\Delta^{n-1}} \rightarrow \mathbb{L}(a_0)$  and we have, by definition,  $\text{Cone}(\mathcal{L}_{\Delta^{n-1}}) = \text{Cyl}(\mathcal{L}_{[0,1]\Delta^{n-1}})/\mathcal{J}$ , where  $\mathcal{J}$  is the ideal generated by the elements  $\hat{a}_i - \hat{a}_0$  and the elements  $\hat{a}_{i_0 \dots i_p}$  for  $p > 0$ .

We define a map,  $\Psi: \mathcal{L}_{\Delta^n} \rightarrow \text{Cyl}(\mathcal{L}_{\Delta^{n-1}})/\mathcal{J}$ , by  $\Psi(a_{i_0 \dots i_p n}) = (-1)^p \overline{a}_{i_0 \dots i_p}$  and  $\Psi(a_n) = \hat{a}_0$ . This map is an isomorphism and, moreover, the quotient  $\text{Cyl}(\mathcal{L}_{\Delta^{n-1}})/\mathcal{J}$  satisfies the three conditions of Proposition 3.1.  $\square$

The next result comes from the restriction to subcomplexes  $X \subset \Delta^n$ .

**Corollary 6.11.** *Let  $X$  be a finite simplicial complex. Then, the cone on  $\mathcal{L}_X$  is isomorphic to the model of the cone on  $X$ ,*

$$\text{Cone}(\mathcal{L}_X) \cong \mathcal{L}_{\text{Cone}(X)}.$$

## 7. TOPOLOGICAL HOMOTOPIES GIVE ALGEBRAIC HOMOTOPIES

The algebraic definition of homotopy respects the topological one.

**Theorem 7.1.** *Let  $f, g: X \rightarrow Y$  be two homotopic morphisms of finite simplicial complexes. Then, their models,  $\mathcal{L}_f, \mathcal{L}_g: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ , are homotopic.*

*Proof.* We give to the product  $X \times [0, 1]$  the structure of a simplicial complex, so that  $X \times \{0\}$  and  $X \times \{1\}$  are simplicial subcomplexes. For  $k = 0, 1$ , we denote by  $j_k$  these canonical inclusions and by  $p: X \times [0, 1] \rightarrow X$  the projection on the first factor. Thus, we have  $\mathcal{L}_p \circ \mathcal{L}_{j_0} = \mathcal{L}_p \circ \mathcal{L}_{j_1} = \text{id}$ .

Now write  $\mathcal{L}_X = (\widehat{\mathbb{L}}(V), d)$  and  $\text{Cyl}(\mathcal{L}_X) = (\widehat{\mathbb{L}}(V \oplus \overline{V} \oplus \widehat{V}), D)$ . Recall that the canonical inclusions of  $\mathcal{L}_X$  in its cylinder are defined by  $\iota_0(v) = v$ ,  $\iota_1(v) = \widehat{v}$ . We can then form the following commutative diagram, in solid arrows,

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{\mathcal{L}_{j_0} \amalg \mathcal{L}_{j_1}} & \mathcal{L}_{X \times [0,1]} \\ \downarrow \iota_0 + \iota_1 & \nearrow \ell & \downarrow \mathcal{L}_p \\ \text{Cyl}(\mathcal{L}_X) & \xrightarrow{q} & \mathcal{L}_X \end{array}$$

where  $q$  is defined by  $q(v) = q(\widehat{v}) = v$  and  $q(\overline{v}) = 0$ . We know from Proposition 6.4 that  $\iota_0 + \iota_1$  is a cofibration. The map  $\mathcal{L}_p$  is clearly surjective; this is also a weak-equivalence, thanks to Lemma 3.4. The lifting property gives a morphism,  $\ell: \text{Cyl}(\mathcal{L}_X) \rightarrow \mathcal{L}_{X \times [0,1]}$  with  $\mathcal{L}_p \circ \ell = q$  and  $\ell \circ \iota_k = \mathcal{L}_{j_k}$  for  $k = 0, 1$ .

Denote by  $h: X \times [0, 1] \rightarrow Y$  the homotopy between  $f$  and  $g$ . The composition

$$\text{Cyl}(\mathcal{L}_X) \xrightarrow{\ell} \mathcal{L}_{X \times [0,1]} \xrightarrow{\mathcal{L}_h} \mathcal{L}_Y$$

is a homotopy between  $\mathcal{L}_f$  and  $\mathcal{L}_g$ . □

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